JOURNAL OF DIFFERENTIAL EQUATIONS **133**, 179–202 (1997)
ARTICLE NO. DE963194

Structure of Positive Radial Solutions of Semilinear Elliptic Equations*

Lynn Erbe and Moxun Tang

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Received November 11, 1994; revised April 22, 1996

We study the positive radial solutions of a semilinear elliptic equation $\Delta u + f(u) = 0$, where $f(u)$ has a supercritical growth order for small $u > 0$ and a subcritical growth order for large u. By showing the uniqueness of positive solutions behaving like $O(|x|^{2-n})$ at infinity, we give an almost complete description for the structure of

1. INTRODUCTION

Let $\mathbb{R}^n$ $(n \geqslant 3)$ denote the usual n-dimensional Euclidean space, and Ω be a finite ball centered at the origin of $\mathbb{R}^n$. We are concerned here with the uniqueness of radial solutions of the problem

$$\begin{aligned} \Delta u + f(u) = 0 \quad & \text{in } \Omega, \\ u > 0 \quad & \text{in } \Omega, \qquad u = 0 \quad \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

or

$$\begin{aligned} \Delta u + f(u) = 0 \quad & \text{in } \mathbb{R}^n, \\ u > 0 \quad & \text{in } \mathbb{R}^n, \qquad u \to 0 \quad \text{as} \quad |x| \to \infty, \end{aligned} \tag{1.2}$$

where f is locally Lipschitz continuous in $[0, \infty)$ and satisfies $f(0) = 0$. Problems (1.1–1.2) arise in various circumstances, for example, in the "fast diffusion" phenomenon in plasma physics (see Berryman and Holland [2]). It is by now well known that all positive solutions of (1.1) are radially symmetric (see Gidas, Ni and Nirenberg [8]).

When $f(u) = u^p$, $1 < p < (n+2)/(n-2)$, it was proved in [8] that problem (1.1) has a unique solution, while problem (1.2) has no radial

* Research supported in part by NSERC-Canada.

0022-0396/97 $25.00

solutions at all. On the other hand, when $f(u)=u^p$, $p \geqslant (n+2)/(n-2)$, we can use the powerful Identity of Pohozaev [25] to demonstrate that there exist no solutions to (1.1), but infinitely many solutions to (1.2). The exponent $(n+2)/(n-2)$ is critical from the point of view of Sobolev embedding. It seems interesting to ask what happens if the nonlinearity f is not a pure power of u, or, if f has both subcritical and supercritical growth in $u>0$.

In the present paper, we shall consider problems (1.1–1.2) for the nonlinearity f which has a supercritical growth for small $u>0$ and a subcritical growth for large u. More precisely, we shall impose the following conditions on $f(u)$:

(f0) $f(0)=0$, and $f(u)>0$ in $u>0$;

(f1) f is superlinear, i.e., $0<f(u)<uf'(u)$ in $u>0$;

(f2) the function $g(u) := uf'(u)/f(u)$ is a decreasing function of $u>0$ and $\lim_{t\to\infty} g(u)<(n+2)/(n-2)$, while $\lim_{t\to 0} g(u)>(n+2)/(n-2)$.

A typical model is

$$f(u)=\begin{cases} u^p & u \geqslant 1, \\ u^q & u<1, \end{cases} \tag{1.3}$$

where $1<p<(n+2)/(n-2)<q$. To deal with the radial solutions of problems (1.1–1.2), we set $t=|x|$, and consider the initial value problem

$$\begin{gathered} u'' + \frac{n-1}{t} u' + f(u) = 0 \qquad t>0, \\ u(0)=\alpha>0, \qquad u'(0)=0. \end{gathered} \tag{1.4}$$

It was proved by Peletier and Serrin [23] that there is a unique solution to (1.4). We shall denote this solution by $u(t, \alpha)$. It is called a *crossing solution* if there exists a $b(\alpha)$, $0<b(\alpha)<\infty$ such that $u(b(\alpha), \alpha)=0$, and $u(t, \alpha)>0$ for $0<t<b(\alpha)$; a *fast decaying solution* if it is positive for all $t>0$ and there is a constant c, $0<c<\infty$ such that $\lim_{t\to\infty} t^{n-2}u(t)=c$; and a *slowly decaying solution* if it is positive for all $t>0$ and $\lim_{t\to\infty} t^{n-2}u(t)=\infty$.

Now we state the main Theorems of this paper for the special case (1.3) to make the essence of our results simple and transparent. On the global structure of the set of solutions of (1.4), our result is

THEOREM 1. *Let f be defined as in (1.3). There exists a unique $1<\alpha^*<+\infty$ such that:*

(i) *If* $0<\alpha<\alpha^*$, *then* $u(t,\alpha)$ *is a slowly decaying solution, and* $\lim_{t\to\infty} t^{2/(q-1)}u(t)=c^*$, *where* $c^*=\{(2/(q-1))(n-2-(2/(q-1)))\}^{1/(q-1)}>0$.

(ii) *If* $\alpha=\alpha^*$, *then* $u(t,\alpha^*)$ *is a fast decaying solution. Moreover,* $u(t,\alpha^*)$ *intersects with* $u(t,\alpha)$, $0<\alpha<\alpha^*$, *exactly once in* $t>0$.

(iii) *If* $\alpha>\alpha^*$, *then* $u(t,\alpha)$ *is a crossing solution. Moreover,* $b(\alpha)$ *is a strictly decreasing function of* α.

On the existence and uniqueness of solutions to problems (1.1–1.2), we have

THEOREM 2. *Let* f *be defined as in* (1.3). *Then we have*

(i) *there exist infinitely many slowly decaying solutions to problem* (1.2);

(ii) *there exists a unique fast decaying radial solution to problem* (1.2);

(iii) *there exists a unique solution to problem* (1.1).

In the main body of this paper we shall devote to prove Theorems 1–2. As we shall see, our proof can be extended to more general case. Namely, the same result of Theorem 2 can be obtained when f satisfies (f0–f2).

The uniqueness problem of (1.1)–(1.2) has been a subject of extensive studies in recent years. When $f(u)$ satisfies (f0)–(f1), Ni [21], and Ni and Nussbaum [22] have given some sufficient conditions for the uniqueness of radial solutions to problem (1.1). More generally, they considered the cases when f may have t-dependence and Ω may be a ball or an annulus. The study of the uniqueness of problem (1.2) can be traced back to Coffman [5] who treated the case when $f(u)=u^3-u$, and $n=3$. His study was continued and generalized by Peletier and Serrin [23, 24], and McLeod and Serrin [18]. In an important development of Kwong [13], it was proved that there is a unique positive solution to $\Delta u-u+u^p=0$ when $1<p<(n+2)/(n-2)$. For other relevant results, see [4, 7, 14–17].

It is worth noting that a similar structure theorem to Theorem 1 was given in Kawano, Yanagida and Yotsutani [12] where they treated the equation $\Delta u+K(|x|)\,u^p=0$, $p>1$, $n\geqslant 3$. A model case of this equation, the so-called Matukuma's equation with $K(|x|)=(1+|x|^2)^{-1}$ and $1<p<(n+2)/(n-2)$ was considered in Yanagida [28]. By an innovative use of the Pohozaev identity, the uniqueness of fast decaying solutions was established. His result was extended in Kwong and Li [14].

The existence of decaying solutions to the equation of (1.1) in $\mathbb{R}^n$ was studied extensively by Kajikiya [9–11]. A model case of the nonlinearities he considered was

$$f(u)=\begin{cases}|u|^{p-1}u & |u|\geqslant 1\\ |u|^{q-1}u & |u|<1,\end{cases} \tag{1.5}$$

where $1 < p < (n+2)/(n-2) < q$. Note that (1.5) reduces to (1.3) when $u > 0$. Specified to (1.5), his result asserts that the equation $\Delta u + f(u) = 0$ has at least one fast decaying radial solution $u(t)$ with $u(0) > 0$ that has exactly k zeros in $0 < t < \infty$, for any given integer $k \geqslant 0$. The uniqueness of these solutions relative to the number of zeros in $(0, \infty)$ remains an open problem. The same problem also arises in Berestycki and Lions [1], McLeod *et al.* [19] and Troy [27]. For the results on the study of the uniqueness of radial solutions with given number of zeros, see Coffman and Marcus [6], Nagasaki [20] and Ni and Nussbaum [22].

We shall employ a shooting argument to prove the main theorems. The remainder of this paper is organized as follows. In Section 2, we shall collect some preliminary results on the general properties of solutions to problem (1.4). In Section 3, we shall show that a fast decaying solution has exactly one intersection point with any slowly decaying solution. Then we can prove that the variational function of the fast decaying solution has exactly one zero in $(0, \infty)$. In Section 4, we shall investigate the asymptotical behavior of the variational function, and prove that any solution $u(t, \alpha)$ with α slightly bigger than the initial value of the fast decaying solution is a crossing solution. In the final section, after proving that $b(\alpha)$ is a strictly decreasing function of α for all large α, we can complete the proof of the main theorems.

2. PRELIMINARY RESULTS

In this section, we collect some preliminary results on the general properties of solutions of (1.4). Throughout the remainder of the paper, we shall assume that condition (f0) is always satisfied. In what follows, we let $u(t)$ or $u(t, \alpha)$ denote the solution of (1.4). For any given $\alpha > 0$, we define $b(\alpha)$ to be the first zero of $u(t, \alpha)$ i.e., $u(b(\alpha), \alpha) = 0$ and $u(t, \alpha) > 0$ for $0 < t < b(\alpha)$. If $u(t, \alpha) > 0$ for all $t > 0$, we simply set $b(\alpha) = \infty$.

PROPOSITION 2.1. *Let $\alpha > 0$, and $u(t, \alpha)$ and $b(\alpha)$ be as above. Then we have*

(i) $u'(t, \alpha) < 0$ *for* $t \in (0, b(\alpha))$.

(ii) *Suppose that* $(f1)$ *holds. Let* $\alpha_1 > 0$, $\alpha_2 > 0$, *and* $\alpha_1 \neq \alpha_2$. *Let* $u_1 = u(t, \alpha_1)$, *and* $u_2 = u(t, \alpha_2)$. *If* u_1 *or* u_2 *is either a crossing solution or a fast decaying solution, then the graphs of* u_1 *and* u_2 *in the* $t - u$ *plane must intersect at least once in* $(0, \min\{b(\alpha_1), b(\alpha_2)\})$.

(iii) (*Pohozaev* [25]) *Let* $F(u) \equiv \int_0^u f(s)\, ds$, *then*

$$\int_0^t \{(n-2)\, uf(u) - 2nF(u)\}\, \tau^{n-1}\, d\tau$$
$$= -(n-2)\, u'(t)\, u(t)\, t^{n-1}\, u'^2(t)\, t^n - 2F(u(t))\, t^n. \tag{2.1}$$

Proof. The identity of (iii) is the well-known Pohozaev Identity (see [25]), we omit its proof here.

(i) Multiply the equation of (1.4) by t^{n-1} to get

$$(t^{n-1}u')' = -t^{n-1}f(u). \tag{2.2}$$

which leads to, for all $t \in (0, b(\alpha))$,

$$t^{n-1}u' = -\int_0^t \tau^{n-1}f(u)\, d\tau < 0.$$

Hence $u'(t, \alpha) < 0$ in $(0, b(\alpha))$.

(ii) Without loss of generality, we may assume $\alpha_2 > \alpha_1$. Suppose to the contrary that the graphs of u_1 and u_2 in the $t-u$ plane are disjointed. Then $u_2 > u_1$ in $0 \leqslant t < b(\alpha_1) = \min\{b(\alpha_1), b(\alpha_2)\}$. It is easy to see that if $b(\alpha_1) < \infty$, then $u_1(\alpha_1, b(\alpha_1)) = 0$ and $u'(\alpha_1, b(\alpha_1)) < 0$. If $u_1 > 0$ in $t > 0$, then $u_2 > u_1 > 0$ for all $t > 0$. We shall show that each case leads to a contradiction.

Recall from (2.2) that

$$(t^{n-1}u_1')' + t^{n-1}f(u_1) = 0,$$
$$(t^{n-1}u_2')' + t^{n-1}f(u_2) = 0.$$

Multiply the first equation by u_2, and the second by u_1, and after subtraction we have

$$(t^{n-1}(u_1'u_2 - u_2'u_1))' = -t^{n-1}(u_2 f(u_1) - u_1 f(u_2)).$$

Integrate this identity over $(0, t)$ to get

$$t^{n-1}(u_1'u_2 - u_2'u_1) = -\int_0^t \tau^{n-1}(u_2 f(u_1) - u_1 f(u_2))\, d\tau. \tag{2.3}$$

Now, if $b(\alpha_1) < \infty$, then we take $t = b(\alpha_1)$ in (2.3) so that the left side is

$$t^{n-1}(u_1'u_2 - u_2'u_1)|_{t=b(\alpha_1)} = b(\alpha_1)^{n-1}\, u_1'(b(\alpha_1)\, u_2(b(\alpha_1))) \leqslant 0.$$

On the other hand, in $(0, b(\alpha_1))$, we have $0 < u_1 < u_2$, which implies $u_2 f(u_1) - u_1 f(u_2) < 0$ in $(0, b(\alpha_1))$ because of (f1). Thus the right side of (2.3) is positive. We obtain a contradiction.

If $u_1 > 0$ in $t > 0$, then $u_2 > u_1 > 0$ in $t > 0$. In this case, u_1 or u_2 is a fast decaying solution, which behaves like t^{2-n} for large t. Hence

$$\lim_{t \to \infty} t^{n-1}(u_1'u_2 - u_2'u_1) = 0.$$

While the same argument as above will show that

$$\int_0^\infty \tau^{n-1}(u_2 f(u_1) - u_1 f(u_2))\, d\tau < 0.$$

We again obtain a contradiction.

In the remainder of this section, we shall recall some results from Kajikiya [10–11] which provide some detailed information of the asymptotic behavior of $u(t, \alpha)$. For the sake of simplicity, we shall restrict the nonlinearity f to be (1.3).

LEMMA 2.2 (Kajikiya [11]). *Let $f(u)$ be defined as in* (1.3). *Suppose that $u(t) = u(t, \alpha) > 0$ in $t > 0$. Then $u(t, \alpha)$ is either a fast decaying solution or a slowly decaying solution. It satisfies exactly one of the following*:

$$\text{(i)} \quad \lim_{t \to \infty} t^{n-2} u(t) = c, \qquad 0 < c < \infty;$$

$$\text{(ii)} \quad \lim_{t \to \infty} t^{2/(q-1)} u(t) = c^*,$$

where c^* is defined as in Theorem 1.

Here we give only an outline of the proof of Lemma 2.2. A detailed proof was given in [11]. Under the assumptions of Lemma 2.2, it was proved in [10] that $u(t)$ satisfies

$$\lim_{t \to \infty} u(t) = \lim_{t \to \infty} u'(t) = 0.$$

Hence, there exists a $T_\alpha > 0$ such that $u(t, \alpha)$ satisfying

$$u'' + \frac{n-1}{t} u' + u^q = 0 \qquad \text{for} \quad t > T_\alpha.$$

A suitable change of variables can be employed to reduce this equation to a certain second order autonomous equation. To complete the proof, the standard phase plane analysis was employed in [11] to investigate the asymptotic behavior of the solutions of the autonomous differential equation.

LEMMA 2.3 (Kajikiya [11]). *Let $f(u)$ be defined as in* (1.3). *If $0 < \alpha \leqslant 1$, then $u(t, \alpha)$ is a slowly decaying solution, i.e., $u(t, \alpha) > 0$ in $t > 0$ and* $\lim_{t \to \infty} t^{2/(q-1)} u(t, \alpha) = c^*$.

This lemma is a part of Lemma 4.3 of [11]. It can be proved by a combination of the Pohozaev Identity and Lemma 2.2. We omit the details here.

3. INTERSECTION PROPERTIES

In this section, we investigate the intersection properties of decaying solutions. The main purpose is to show that a fast decaying solution intersect each slowly decaying solution exactly once in $t > 0$. This result will be stated in Lemma 3.3, which is crucial in the next section in studying the oscillatory and asymptotic behavior of the variational function of the fast decaying solution.

To make our presentation simpler and more transparent, in the remainder of this paper we shall assume that the nonlinearity f is given by (1.3), unless otherwise specified. But in a few remarks, we shall briefly explain how our proof can be extended to more general cases.

Define:

$N := \{\alpha \mid \alpha > 0, u(t, \alpha) \text{ has a finite zero } b(\alpha) \text{ in } (0, \infty)\}$.

$D_f :=\{\alpha \mid \alpha > 0, u(t, \alpha) \text{ is a positive and fast decaying solution in } (0, \infty)\}$.

$D_s :=\{\alpha \mid \alpha > 0, u(t, \alpha) \text{ is a positive and slowly decaying solution } (0, \infty)\}$.

$D := D_f \cup D_s$.

By Lemma 2.2, we see that $N \cup D = (0, \infty)$.

LEMMA 3.1. *The sets N, D_s are open in $(0, \infty)$, and D_f is closed.*

Proof. That N is open is a simple consequence of continuous dependence of solutions of initial value problems for ordinary differential equations. It remains to prove that D_s is open.

For any $\alpha \in D$, it follows from Lemma 2.2 that there exists some $T_\alpha > 0$ such that $0 < u(t, \alpha) < 1$ for all $t > T_\alpha$, and then $u(t, \alpha)$ satisfies

$$u'' + \frac{n-1}{t} u' + u^q = 0 \qquad t \in (T_\alpha, \infty), \tag{3.1}$$

Now we employ the change of variables as in [11]. Let

$$x(s) = t^\beta u(t), \qquad t = e^s, \qquad \beta = \frac{2}{q-1}. \tag{3.2}$$

Then (3.1) is transformed into an autonomous differential equation

$$x'' + ax' - bx + x^q = 0 \qquad s \in (\log T_\alpha, \infty), \tag{3.3}$$

where $a = n-2-2\beta(>0)$, $b = \beta(n-2-\beta)(>0)$. Define

$$E(u(t,\alpha)) \equiv E(u,t) \equiv E(x(s,\alpha)) \equiv E(x,s)$$
$$\equiv \frac{1}{2}x'^2(s) + \frac{1}{q+1}x(s)^{q+1} - \frac{b}{2}x(s)^2, \tag{3.4}$$

then

$$\frac{dE(x,s)}{ds} = -ax'(s)^2 \leqslant 0. \tag{3.5}$$

Since the zeros of $x'(s)$ are isolated, it follows that $E(x,s)$ is a strictly decreasing function of s.

Note that for any $\alpha \in D_s$, we have $x(s) \to c^*$, $x'(s) \to 0$ as $s \to \infty$, so then,

$$\lim_{t\to\infty} E(u,t) = \lim_{s\to\infty} E(x,s) = \frac{1}{q+1}(c^*)^{q+1} - \frac{b}{2}c^{*2} = d < 0, \tag{3.6}$$

where

$$d = -\frac{q-1}{2(q+1)} b^{(q+1)/(q-1)} < 0. \tag{3.7}$$

For any $\alpha \in D_f$, we have $x(s) \to 0$, $x'(s) \to 0$, so

$$\lim_{t\to\infty} E(u,t) = \lim_{t\to\infty} E(x,s) = 0. \tag{3.8}$$

While if $\alpha \in N$, then $b(\alpha) < \infty$ and

$$E(u, b(\alpha)) = E(x, \log b(\alpha)) \geqslant 0. \tag{3.9}$$

Now we fix an $\alpha_0 \in D_s$, and find a $T_1 > T_\alpha$ such that

$$E(u(T_1, \alpha_0)) < -\frac{d}{2}.$$

If $\varepsilon > 0$ is sufficiently small, and $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, then we have

$$0 < u(T_1, \alpha) < 1, \qquad \text{and} \qquad E(u(T_1, \alpha)) < -\frac{d}{4}. \tag{3.10}$$

Combining (3.5), (3.9) and (3.10) we see that $u(t,\alpha)$ is not a crossing solution. Since $E(u(t,\alpha))$ is monotonically decreasing, we have

$$\lim_{t\to\infty} E(u(t,\alpha)) \leqslant -\frac{d}{4} < 0.$$

Hence $\alpha \notin D_f$, which implies $\alpha \in D_s$ and D_s is an open set. The proof is completed.

Define

$$A := \{\alpha \mid \alpha > 0, u(t, \alpha') \text{ is a slowly decaying solution for every } \alpha' \in (0, \alpha)\}. \tag{3.11}$$

The set A is nonempty, in fact $(0, 1) \subset A$. That A is bounded above follows from the existence of crossing solutions. In fact, it is proved that crossing solutions exist if $f(u)$ has a subcritical growth in a neighborhood of infinity, (see [3, 9] and [26]). So we can define:

$$\bar{\alpha} := \sup \{\alpha \mid \alpha \in A\}. \tag{3.12}$$

As a consequence of Lemma 3.1, we conclude that $\bar{\alpha} \in D_f$. By Lemma 2.3, we have $\bar{\alpha} > 1$.

Next we shall show that $u(t, \bar{\alpha})$ intersects $u(t, \alpha)$, $\alpha \in A$ exactly once in $(0, \infty)$. Note that in Proposition (ii) we proved that they intersect at least once.

LEMMA 3.2. *There exists a sufficiently small* $\varepsilon > 0$ *such that* $u(t, \bar{\alpha})$ *intersects* $u(t, \alpha)$ *exactly once in* $(0, \infty)$ *for each* $\alpha \in (0, \varepsilon)$.

Proof. Suppose the contrary. Then there exists a sequence $\{\alpha_i\}_{i=1}^{\infty}$, $\alpha_i < 1$, $\alpha_i \to 0$ as $i \to \infty$ such that $u(t, \alpha)$ intersects each $u(t, \alpha_i)$ at least twice. Let the first two intersection points of $u(t, \bar{\alpha})$ with $u(t, \alpha_i)$ occur at $t = a_i$, b_i, $a_i < b_i$. Then we have

$$\lim_{i \to \infty} a_i = \lim_{i \to \infty} b_i = +\infty. \tag{3.13}$$

Set

$$L(\alpha, t) \equiv -(n-2)\, u'(t, \alpha)\, u(t, \alpha)\, t^{n-1} - u'(t, \alpha)^2\, t^n - 2F(u(t, \alpha))\, t^n, \tag{3.14}$$

$$P(\alpha, t) \equiv (n-2)\, u(\alpha, t)\, f(u(\alpha, t)) - 2nF(u(\alpha, t)). \tag{3.15}$$

Then the Pohozaev identity (2.1) becomes

$$\int_0^t P(\alpha, \tau)\, \tau^{n-1}\, d\tau = L(\alpha, t). \tag{3.16}$$

Since $\alpha_i < 1$, we have $u(t, \alpha_i) < 1$ for all $t \in (0, \infty)$, and so

$$P(\alpha_i, t) = \left(n - 2 - \frac{2n}{q+1}\right) u^{q+1} > 0.$$

By (3.16) we have

$$L(\alpha_i, t) > 0 \qquad \text{for all} \quad t \in (0, \infty). \tag{3.17}$$

On the other hand, recall from Lemma 2.2 that $t^{n-2}u(t\bar{\alpha})$ tends to a finite number as $t \to \infty$. Therefore,

$$L(\bar{\alpha}, \infty) \equiv \lim_{t \to \infty} L(\bar{\alpha}, t) = 0, \tag{3.18}$$

Let t_1 be the unique number such that $u(t_1, \bar{\alpha}) = 1$. If $t > t_1$, then $u(t, \bar{\alpha}) < 1$, and so $P(\bar{\alpha}, t) > 0$ and

$$L(\bar{\alpha}, t) = -\int_t^{\infty} P(\bar{\alpha}, \tau)\, \tau^{n-1}\, d\tau < 0. \tag{3.19}$$

Now we compare $L(\bar{\alpha}, t)$ with $L(\alpha_i, t)$ at the second intersection point $t = b_i$. If i is sufficiently large, then by (3.17) and (3.19) we simply get

$$L(\bar{\alpha}, b_i) < L(\alpha_i, b_i). \tag{3.20}$$

Since $u(b_i, \bar{\alpha}) = u(b_i, \alpha_i)$, we have:

$$\begin{aligned} &-(n-2)\, u'(b_i, \bar{\alpha})\, u(b_i, \bar{\alpha})\, b_i^{n-1} - u'(b_i, \bar{\alpha})^2\, b_i^n \\ &\quad < -(n-2)\, u'(b_i, \alpha_i)\, u(b_i, \alpha_i)\, b_i^{n-1} - u'(b_i, \alpha_i)^2\, b_i^n, \end{aligned} \tag{3.21}$$

that is

$$\begin{aligned} &(n-2)(u'(b_i, \bar{\alpha}) - u'(b_i, \alpha_i))\, u(b_i, \bar{\alpha})\, b_i^{n-1} \\ &\quad + (u'(b_i, \bar{\alpha})^2 - u'(b_i, \alpha_i)^2)\, b_i^n > 0. \end{aligned} \tag{3.22}$$

Note that both $u(t, \bar{\alpha})$ and $u(t, \alpha_i)$ are decreasing in $(0, \infty)$, and since $\alpha_i < \bar{\alpha}$, we see that at the second intersection point we have

$$u'(b_i, \alpha_i) < u'(b_i, \bar{\alpha}) < 0. \tag{3.23}$$

Combining (3.22) and (3.23) we get

$$(n-2)\, u(b_i, \bar{\alpha})\, b_i^{n-1} + (u'(b_i, \bar{\alpha}) + u'(b_i, \alpha_i))\, b_i^n > 0, \tag{3.24}$$

and this in turn, with (3.23), implies

$$(n-2)\, u(b_i, \bar{\alpha})\, b_i^{n-1} + 2u'(b_i, \bar{\alpha})\, b_i^n > 0,$$

so then

$$(n-2)\, u(b_i, \bar{\alpha})\, b_i^{n-2} + 2u'(b_i, \bar{\alpha})\, b_i^{n-1} > 0. \tag{3.25}$$

We shall show that (3.25) contradicts the fact that $u(t, \bar{\alpha})$ is a fast decaying solution. Let $0 < c < \infty$ be such that

$$\lim_{t \to \infty} u(t, \bar{\alpha})\, t^{n-2} = c > 0. \tag{3.26}$$

It follows from (2.2) that $t^{n-1} u'(t, \bar{\alpha})$ is a decreasing function of t. Thus $\lim_{t \to \infty} t^{n-1} u'(t, \bar{\alpha})$ exists. By L'Hospital's rule we have

$$\begin{aligned} c &= \lim_{t \to \infty} t^{n-2} u(t, \bar{\alpha}) = \lim_{t \to \infty} \frac{u(t, \bar{\alpha})}{t^{2-n}} = \lim_{t \to \infty} \frac{u'(t, \bar{\alpha})}{(2-n)\, t^{n-1}} \\ &= \frac{1}{2-n} \lim_{t \to \infty} t^{n-1} u'(t, \bar{\alpha}), \end{aligned}$$

which leads to

$$\lim_{t \to \infty} t^{n-1} u'(t, \bar{\alpha}) = (2-n)\, c < 0. \tag{3.27}$$

Combining (3.26), (3.27) we obtain:

$$\begin{aligned} &\lim_{t \to \infty} (n-2)\, t^{n-2} u(t, \bar{\alpha}) + 2t^{n-1} u'(t, \bar{\alpha}) \\ &\qquad = (n-2)\, c + 2(2-n)\, c = (2-n)\, c < 0. \end{aligned} \tag{3.28}$$

Since $b_i \to \infty$ as $i \to \infty$, it is evident that (3.25) contradicts (3.28). This proves the lemma.

Lemma 3.3. *$u(t, \bar{\alpha})$ intersects $u(t, \alpha)$ exactly once in $(0, \infty)$ for any $\alpha \in A$.*

Proof. Suppose this lemma is not true. Define a subset A_1 of A by

$$A_1 := \{\alpha \mid \alpha \in A,\ u(t, \bar{\alpha}) \text{ intersects } u(t, \alpha) \text{ at least twice in } (0, \infty)\} \tag{3.29}$$

By the assumption, A_1 is not empty. Since $(0, \varepsilon) \cap A_1 = \emptyset$, so $A_1 \neq A$, where ε is from Lemma 3.2.

Define

$$\hat{\alpha} := \inf\{\alpha \mid \alpha \in A_1\}. \tag{3.30}$$

Then $\hat{\alpha} \geqslant \varepsilon > 0$. Note that A_1 is an open subset of A, which follows from the continuous dependence of solutions of initial value problems for ordinary differential equations. Also, A is an open subset of $(0, \infty)$, which in turn implies that A_1 is an open subset of $(0, \infty)$. So we have

$$\hat{\alpha} \notin A_1.$$

By the definition of $\hat{\alpha}$ in (3.30), there exists a sequence $\{\alpha_j\}_{j=1}^{\infty}$ such that $\alpha_j \in A$ and $\lim_{j\to\infty} \alpha_j = \hat{\alpha}$. Let $\{e_j\}_{j=1}^{\infty}$ be a sequence of values of t at which $u(t, \bar{\alpha})$ and $u(t, \alpha_j)$ have the second intersection point. Then

$$\lim_{j\to\infty} e_j = \infty. \tag{3.31}$$

Now we turn to consider equation (3.3). Without loss of generality, we can suppose that for all e_j, $j = 1, 2, ..., n$, it holds that $u(e_j, \bar{\alpha}) < 1$. Let $x(s, \alpha)$ be as in (3.2). Then $x(s, \bar{\alpha})$ intersects $x(s, \alpha_j)$ at $s_j = \log e_j$. Let d be as in (3.7), and choose $\hat{T}$ sufficiently large such that

$$E(x(\hat{T}, \hat{\alpha})) < \frac{d}{2} < 0, \tag{3.32}$$

and

$$x(x, \bar{\alpha}) \text{ is decreasing for } s > \hat{T}. \tag{3.33}$$

Note that (3.32) follows from (3.6) and (3.33) follows from Lemma 2.2. Now we choose a subsequence of $\{\alpha_j\}_{j=1}^{\infty}$, and for simplicity of notation, we still denote it by $\{\alpha_j\}_{j=1}^{\infty}$, such that

$$s_j > \hat{T}, \qquad \text{and} \qquad E(x(\hat{T}, \alpha_j)) < \frac{d}{4} < 0, \qquad \text{for all } j.$$

By (3.5) we find that

$$E(x(s_j, \alpha_j)) < \frac{d}{4} < 0 \qquad \text{for all } j, \tag{3.34}$$

Since $x(s, \bar{\alpha})$ is decreasing at s_j at which $x(s, \bar{\alpha})$ intersects with $x(s, \alpha_j)$ for the second time, we obtain

$$x'(s_j, \alpha_j) < x'(s_j, \bar{\alpha}) < 0,$$

or

$$x'(s_j, \alpha_j)^2 > x'(s_j, \bar{\alpha})^2 > 0.$$

By (3.6), we have

$$E(x(s_j, \alpha_j)) > E(x(s_j, \bar{\alpha})). \tag{3.35}$$

Combining (3.31), (3.34) and (3.35), we get

$$\liminf_{s \to \infty} E(x(s, \bar{\alpha})) \leq \frac{d}{4} < 0.$$

But this contradicts the fact that $\lim_{s \to \infty} E(x(s, \bar{\alpha}), \bar{\alpha}) = 0$. The proof is completed.

Remark 3.4. In the general case when $f(u)$ is a given nonlinearity satisfying (f1–f2), the conclusions of Lemmas 3.1–3.3 hold if $\bar{\alpha}$ is defined in a similar way to (3.12). The proof of Lemma 3.2 in the general case is exactly the same as above. Lemma 3.1 and Lemma 3.2 can be established with a slightly modified approach. We may make use of an "energy function" argument by introducing

$$M(t, \alpha) = t^{2n-2}[u'^2(t, \alpha)/2 + F(u(t, \alpha))],$$

instead of $E(u(\alpha, t))$.

4. CROSSING SOLUTIONS NEAR $u(t, \bar{\alpha})$

The main purpose of this section is to show that all the solutions with initial value α slightly bigger than $\bar{\alpha}$ must be crossing solutions. As in [7, 13], and [16, 17], we employ a Kolodner-Coffman method. For a given solution $u(t, \alpha)$, define its variational function as

$$\delta(t, \alpha) \equiv \frac{\partial u(t, \alpha)}{\partial \alpha}, \tag{4.1}$$

which satisfies

$$\begin{gathered} \delta'' + \frac{n-1}{t}\delta' + f'(u)\,\delta = 0 \\ \delta(0) = 1, \qquad \delta'(0) = 0. \end{gathered} \tag{4.2}$$

It follows from lemma 3.3 that $\delta(t, \bar{\alpha})$ vanishes at most once in $(0, \infty)$. In the next lemma, we prove that $\delta(t, \bar{\alpha})$ must vanish at least once.

LEMMA 4.1. *Let $u(t, \alpha)$ be a fast decaying solution or a crossing solution. Then $\delta(t, \alpha)$ vanishes at least once in $(0, b(\alpha))$, where $b(\alpha)$ may be $+\infty$.*

Proof. From (1.4) and (4.2) we get

$$(t^{n-1}u')' = -t^{n-1}f(u), \tag{4.3}$$

$$(t^{n-1}\delta')' = -t^{n-1}f'(u)\,\delta. \tag{4.4}$$

Multiply (4.3), (4.4) by δ, u, respectively and after subtraction we have

$$(t^{n-1}(u'\delta - u\delta'))' = -t^{n-1}\,\delta(f(u) - uf'(u)). \tag{4.5}$$

Now if $u(t, \alpha)$ is a crossing solution with $0 < b(\alpha) < +\infty$, and $\delta(t, \alpha)$ does not vanish in $(0, b(\alpha))$, then

$$u(t, \alpha) > 0 \quad \text{for} \quad t \in (0, b(\alpha)), \qquad u(b(\alpha), \alpha) = 0, \qquad u'(b(\alpha), \alpha) < 0, \tag{4.6}$$

and

$$\delta(t, \alpha) \geqslant 0 \quad \text{for} \quad t \in (0, b(\alpha)), \text{ where the equality may hold only at } b(\alpha). \tag{4.7}$$

Let $t = b(\alpha)$ in (4.5), we obtain

$$b(\alpha)^{n-1}u'(b(\alpha))\,\delta(b(\alpha)) = -\int_0^{b(\alpha)} t^{n-1}\delta(t)(f(u) - uf'(u))\,dt.$$

By (4.6) and (4.7) we see that the left side is nonpositive. While the right side is positive due to (4.7) and the fact that $f(u) - uf'(u) < 0$, $t \in (0, b(\alpha))$, yielding a contradiction.

Next we consider the case when $u(t, \alpha)$ is a fast decaying solution. Suppose that $\delta(t, \alpha)$ never vanishes in $(0, \infty)$. Then

$$\delta(t, \alpha) > 0 \quad \text{for all} \quad t \in (0, \infty). \tag{4.8}$$

By (4.4) we see that

$$\lim_{t \to \infty} t^{n-1}\delta' = c_1 < 0 \text{ exists, where } c_1 \text{ may be } -\infty. \tag{4.9}$$

By (4.4) and (4.8) we see that $\delta(t, \alpha)$ is decreasing. Integrating both sides of (4.5) we obtain

$$t^{n-1}(u'\delta - u\delta') = -\int_0^t \tau^{n-1}\delta(f(u) - uf'(u))\,d\tau > 0, \qquad t \in (0, \infty). \tag{4.10}$$

Therefore $u'\delta - u\delta' > 0$, $t \in (0, \infty)$, which implies $u(t, \alpha)/\delta(t, \alpha)$ is strictly increasing in $(0, \infty)$. Thus

$$\frac{u(t, \alpha)\, t^{n-2}}{\delta(t, \alpha)\, t^{n-2}} \text{ is strictly increasing in } (0, \infty). \tag{4.11}$$

Since $\lim_{t \to \infty} u(t, \alpha)\, t^{n-2} = c > 0$, there is a c_∞ such that

$$\lim_{t \to \infty} \frac{u(t, \alpha)\, t^{n-2}}{\delta(t, \alpha)\, t^{n-2}} = c_\infty. \tag{4.12}$$

If $c_\infty < +\infty$, then $\lim_{t \to \infty} \delta(t, \alpha)\, t^{n-2} = c/c_\infty < +\infty$. By the L'Hospital's rule and (4.9), we have

$$\lim_{t \to \infty} \delta(t, \alpha)\, t^{n-2} = \lim_{t \to \infty} \frac{\delta(t, \alpha)}{t^{2-n}} = \frac{1}{2-n} \lim_{t \to \infty} \delta'(t, \alpha)\, t^{n-1}.$$

Thus

$$\lim_{t \to \infty} \delta'(t, \alpha)\, t^{n-1} = (2-n)\, \frac{c}{c_\infty}. \tag{4.13}$$

If $c_\infty = +\infty$, then we get

$$\lim_{t \to \infty} \delta'(t, \alpha)\, t^{n-1} = 0. \tag{4.14}$$

Combining (4.9), (4.13–4.14) and from that $\lim_{t \to \infty} t^{n-1} u' = (2-n)\, c < 0$, $\lim_{t \to \infty} u(t, \alpha) = 0$, and $\lim_{t \to \infty} \delta(t, \alpha) \equiv c_3 \geqslant 0$ (which exists since $\delta(t, \alpha)$ is monotonically decreasing), we have

$$\begin{aligned}\lim_{t \to \infty} t^{n-1}(u'\delta - u\delta') &= \lim_{t \to \infty} (t^{n-1}u')\, \delta - \lim_{t \to \infty} u(t^{n-1}\delta') \\ &= (2-n)\, cc_2 \leqslant 0.\end{aligned}$$

On the other hand, by (4.5), we have $\lim_{t \to \infty} t^{n-1}(u'\delta - u\delta') > 0$, a contradiction. The proof is completed.

Lemma 3.3 and Lemma 4.1 imply that $\delta(t, \bar{\alpha})$ has a unique zero, say, $t = \tau$ in $(0, \infty)$, and

$$\delta(t, \bar{\alpha}) > 0 \quad \text{for} \quad t \in (0, \tau), \qquad \delta(\tau, \bar{\alpha}) = 0, \qquad \delta(t, \bar{\alpha}) < 0 \quad \text{for} \quad t > \tau. \tag{4.15}$$

In the following, we shall show that there are some numbers $\lambda > 0$ such that $t^\lambda \delta(t, \bar{\alpha}) \to -\infty$, as $t \to \infty$. Before doing this, we give an identity involving $u(t)$ and $\delta(t)$. This identity was used in [7, 16] and can be verified by routine calculation, so we omit its proof.

LEMMA 4.2. *Let u, δ be the solutions of* (1.4), (4.2) *respectively. Then*

$$[t^{n-1}(tu)''\,\delta-(tu)'\,\delta']'=-t^{n-1}\delta[3f(u)-uf'(u)]. \tag{4.16}$$

Define

$$w\equiv w_\lambda(t,\alpha)\equiv w_\lambda(t)=t^\lambda u(t,\alpha). \tag{4.17}$$

Then w satisfies the equation

$$w''+(n-1-2\lambda)\frac{w'}{t}+\lambda(\lambda+2-n)\frac{w}{t^2}+t^\lambda f(u)=0. \tag{4.18}$$

Note that if

$$\frac{2}{q-1}<\lambda<n-2. \tag{4.19}$$

Then

$$\lim_{t\to\infty} w_\lambda(t,\alpha)=0 \qquad \text{if } u(t,\alpha) \text{ is a fast decaying solution,} \tag{4.20}$$

and

$$\lim_{t\to\infty} w_\lambda(t,\alpha)=\infty \qquad \text{if } u(t,\alpha) \text{ is a slowly decaying solution.} \tag{4.21}$$

Let

$$y\equiv y_\lambda(t,\alpha)\equiv\frac{\partial w_\lambda(t,\alpha)}{\partial\alpha}=\frac{\partial t^\lambda(t,\alpha)}{\partial\alpha}=t^\lambda\,\frac{\partial u(t,\alpha)}{\partial\alpha}=t^\lambda\delta(t,\alpha). \tag{4.22}$$

Then y satisfies the equation

$$y''+(n-1-2\lambda)\frac{y'}{t}+\lambda(\lambda+2-n)\frac{y}{t^2}+f'(u)\,y=0. \tag{4.23}$$

Note that $y(t,\bar{\alpha})$ has the same zero point as $\delta(t,\bar{\alpha})$, and (4.15) holds if δ is replaced by y. Let λ^* be defined by

$$\lambda^*=\begin{cases}\dfrac{n-1}{2} & \text{if } \ n>3\\[2ex] \dfrac{1}{2} & \text{if } \ n=3,\end{cases} \tag{4.24}$$

then λ^* satisfies (4.19). When $n>3$, $y_{\lambda^*}(t,\bar{\alpha})$ satisfies

$$y''+\left(\frac{(n-1)(3-n)}{4t^2}+f'(u)\right)y=0. \tag{4.25}$$

By (3.26), we have $u(t,\bar{\alpha})\sim ct^{2-n}$ as $t\to\infty$, and this implies,

$$f'(u)=qu^{q-1}\sim qc^{q-1}t^{(2-n)(q-1)} \qquad \text{as} \quad t\to\infty. \tag{4.26}$$

But $(2-n)(q-1)<-4$, since $q>(n+2)/(n-2)$, so the coefficient of the second term in (4.25) is eventually negative, and in fact, for some sufficiently large $T>0$ we have

$$y_{\lambda^*}<0, \qquad y''_{\lambda^*}<0, \qquad \text{and}$$
$$\frac{(n-1)(3-n)}{2t^2}<\frac{(n-1)(3-n)}{4t^2}+f'(u)<\frac{(n-1)(3-n)}{8t^2} \qquad \text{for} \quad t>T. \tag{4.27}$$

Thus $y_{\lambda^*}(t,\bar{\alpha})$ is eventually monotone, so we can define

$$y_\infty\equiv\lim_{t\to\infty} y_{\lambda^*}(t,\bar{\alpha}). \tag{4.28}$$

Obviously $-\infty\leqslant y_\infty\leqslant 0$, and

$$\begin{aligned}&\text{(i)}\quad \text{if } y_\infty>-\infty,\text{ then } y'_{\lambda^*}(t,\bar{\alpha})\to 0 \text{ as } t\to\infty;\\ &\text{(ii)}\quad \text{if } y_\infty=-\infty,\text{ then } y'_{\lambda^*}(t,\bar{\alpha})<0 \text{ for large } t.\end{aligned} \tag{4.29}$$

When $n=3$, we have, by substituting $\lambda^*=\frac{1}{2}$, $n=3$ into (4.23)

$$y''+\frac{y'}{t}-\frac{y}{4t^2}+f'(u)\ y=0. \tag{4.30}$$

That is

$$(ty')+t\left(f'(u)-\frac{1}{4t^2}\right)y=0. \tag{4.31}$$

Since $u(t,\bar{\alpha})\sim c\cdot t^{-1}$ as $t\to\infty$, we have $f'(u)\sim qc^{q-1}t^{1-q}$. Since $1-q<-4$, the coefficient of the second term in (4.31) is also negative for large t. In fact, a similar inequality to (4.27) holds. As a consequence, ty'_{λ^*} is eventually monotonically decreasing, so then

$$\lim_{t\to\infty} ty'_{\lambda^*}(t,\bar{\alpha})\equiv l \text{ exists}, \qquad -\infty\leqslant l<+\infty.$$

If $l > -\infty$, we have

$$\lim_{t \to \infty} t^{1/2} y'_{\lambda^*}(t, \bar{\alpha}) = 0. \tag{4.32}$$

On the other hand, if $l = -\infty$, then for some sufficiently large t_0 we have $ty'_{\lambda^*}(t, \bar{\alpha}) < -1$ for $t \geqslant t_0$, and, if $t > t_0$,

$$\begin{aligned} y_{\lambda^*}(t, \bar{\alpha}) &= \int_{t_0}^{t} y'_{\lambda^*}(t, \bar{\alpha})\, dt + y_{\lambda^*}(t_0, \bar{\alpha}) \\ &= \int_{t_0}^{t} \frac{ty'_{\lambda^*}(t, \bar{\alpha})}{t}\, dt + y_{\lambda^*}(t_0, \alpha) \\ &< -\int_{t_0}^{t} \frac{1}{t}\, dt + y_{\lambda^*}(t_0, \bar{\alpha}). \end{aligned}$$

Letting t tend to ∞, we get $\lim_{t \to \infty} y(t, \bar{\alpha}) = -\infty$.

Note that ty'_{λ^*} is eventually monotone, so y_{λ^*} is eventually monotone and we can define y_∞, as in the case $n > 3$, to be the limit of $y_{\lambda^*}(t, \bar{\alpha})$ as $t \to \infty$. We also have $-\infty < y_\infty \leqslant 0$. If $y_\infty > -\infty$, then $l > -\infty$ and (4.32) holds. Similar to (4.29), we have

$$\begin{array}{llll} \text{(i)} \;\; \text{if} \;\; y_\infty > -\infty, & \text{then} \;\; t^{1/2} y'_{\lambda^*}(t, \bar{\alpha}) \to 0 & \text{as} \;\; t \to \infty; \\ \text{(ii)} \;\; \text{if} \;\; y_\infty = -\infty, & \text{then} \;\; y'_{\lambda^*}(t, \bar{\alpha}) < 0 & \text{for} \;\; \text{large } t. \end{array} \tag{4.33}$$

In the next lemma we shall show that case (i) of (4.29) and (4.33) cannot happen.

LEMMA 4.3. *If λ^* is defined by* (4.24), *then $y_{\lambda^*}(t, \bar{\alpha})$ approaches $-\infty$ as $t \to \infty$.*

Proof. We prove this lemma by employing identities (4.5) and (4.16). After integrating both identities over $(0, \infty)$, we get

$$\lim_{t \to \infty} t^{n-1}(u'\delta - u\delta') = -\int_0^\infty t^{n-1} \delta(f(u) - uf'(u))\, dt, \tag{4.34}$$

$$\lim_{t \to \infty} t^{n-1}((tu)''\delta - (tu)'\, \delta') = -\int_0^\infty t^{n-1} \delta(3(u) - uf'(u))\, dt. \tag{4.35}$$

If $n > 3$ and suppose to the contrary that case (i) of (4.29) occurs. Then $t^{(n-1)/2}\delta(t, \bar{\alpha}) \to y_\infty > -\infty$, and $(t^{(n-1)/2}\delta(t, \bar{\alpha}))' \to 0$ as $t \to \infty$, which imply (recall that $(n-1)/2 > 1$),

$$\delta(t, \bar{\alpha}) \to 0 \qquad t\delta'(t, \bar{\alpha}) \to 0 \qquad \text{as} \quad t \to \infty. \tag{4.36}$$

Recall that $t^{n-1}u'(t, \bar{\alpha})$ and $t^{n-2}u(t, \bar{\alpha})$ approach some finite constants as $t \to \infty$. We have

$$\lim_{t \to \infty} t^{n-1}(u'\delta - u\delta') = \lim_{t \to \infty} (t^{n-1}u')\,\delta - \lim_{t \to \infty} (t^{n-1}u)(t\delta') = 0,$$

and

$$\lim_{t \to \infty} t^{n-1}((tu)''\delta - (tu)'\,\delta')$$
$$= \lim_{t \to \infty} ((3-n)(t^{n-1}u')\,\delta - t^n f(u)\,\delta - (t^{n-1}u)(t\delta') - (t^{n-1}u')(t\delta')) = 0.$$

(Note that $t^n f(u)$ approaches zero since eventually its growth order is less than -2). These identities imply

$$\int_0^\infty t^{n-1}\delta(f(u) - uf'(u))\,dt = 0,$$

$$\int_0^\infty t^{n-1}\delta(3f(u) - uf'(u))\,dt = 0.$$

So we have

$$\int_0^\infty t^{n-1}\delta(pf(u) - uf'(u))\,dt = 0, \tag{4.37}$$

$$\int_0^\infty t^{n-1}\delta(qf(u) - uf'(u))\,dt = 0. \tag{4.38}$$

But, in any case, at least one of (4.37), (4.38) cannot be true. For instance, if $u(\tau) > 1$, and τ is defined as in (4.15), letting t_1 be the unique value such that $u(t_1, \bar{\alpha}) = 1$, then

$$\int_0^\infty t^{n-1}\delta(pf(u) - uf'(u))\,dt = \int_{t_1}^\infty t^{n-1}\delta(p-q)\,u^q\,dt > 0.$$

So we get a contradiction, and the lemma is proved for $n > 3$. If $n = 3$, then the same argument still works, since case (i) of (4.33) will also imply (4.36).

Now we can obtain our main result of this section, which can be accomplished by showing.

LEMMA 4.4. *There exists a sufficiently small number $\gamma > 0$ such that any solution $u(t, \alpha)$ with initial height $\alpha \in (\bar{\alpha}, \bar{\alpha} + \gamma]$ is a crossing solution.*

Proof. We prove this lemma for $n > 3$. In case $n = 3$, the proof is similar, and we omit it. Recall that $\delta(t, \bar{\alpha})$ has a unique zero in $(0, \infty)$. It follows that there exist constants $\gamma_0 > 0$ sufficiently small, and $T_0 > T > 0$ sufficiently large (T is defined in (4.27)), such that for each $\alpha \in (\bar{\alpha}, \bar{\alpha} + \gamma_0]$, $u(t, \alpha)$ intersects $u(t, \bar{\alpha})$ exactly once in $(0, T_0)$.

Now we choose $\alpha_1 \in (\bar{\alpha}, \bar{\alpha} + \gamma_0]$, α_1 sufficiently close to α. Define

$$w_1 = w_{\lambda *}(t, \alpha_1) = t^{(n-2)/2} u(t, \alpha_1),$$

and write $\bar{w} = t^{(n-1)/2} u(t, \bar{\alpha})$. Let

$$v = w_1 - \bar{w}. \tag{4.39}$$

In view of (4.27) and (ii) of (4.29), we may assume $v(T_0) < 0$, $v'(T_0) < 0$. Note that v satisfies

$$v'' + \frac{(n-1)(3-n)}{4t^2} v + t^{(n-2)/2}(f(u(t, \alpha_1)) - f(u(t, \bar{\alpha}))) = 0.$$

That is,

$$v'' + \left(\frac{(n-1)(3-n)}{4t^2} + f'(\xi(t)) \right) v = 0, \qquad \xi(t) \in (u(t, \alpha_1), u(t, \bar{\alpha})). \tag{4.40}$$

If $u(t, \alpha_1)$ is a slowly (fast) decaying positive solution, we have $w_1 \to \infty$ ($w_1 \to 0$) by (4.20) or (4.21). In both cases, we can find some $T_1 > T_0$ such that

$$v(T_1) < 0, \qquad v'(T_1) = 0, \qquad v(t) \text{ has a local minimum at } t = T_1. \tag{4.41}$$

But, on the other hand, by (4.27) and (4.40), we have

$$v''(T_1) < 0,$$

which leads to a contradiction. This shows that $u(t, \alpha_1)$ is a crossing solution. The proof is completed.

Remark 4.5. More generally, if $f(u)$ is not given by (1.3), then the conclusions of Lemmas 4.1–4.4 also hold under assumptions (f1)–(f2) with the same proof.

5. PROOF OF THEOREM 1

In this section, we shall complete the proof of the main results. At first, we state and prove two lemmas which assert that the variational functions of a crossing solution has exactly one zero in $(0, b(\alpha))$. Recall that in Lemma 4.4 we have shown that $u(t, \alpha)$ is a crossing solution if $\alpha \in (\bar{\alpha}, \bar{\alpha}+\gamma]$.

LEMMA 5.1. *There exists $\alpha \in (\bar{\alpha}, \bar{\alpha}+\gamma]$ such that the variational function $\delta(t, \alpha)$ has exactly one zero in $(0, b(\alpha))$.*

Proof. By Lemma 4.1, $\delta(t, \alpha)$ must vanish at least once in $(0, b(\alpha))$, for all $\alpha \in (\bar{\alpha}, \bar{\alpha}+\gamma]$. If the assertion if Lemma 5.1 is not true, then for any $\alpha \in (\bar{\alpha}, \bar{\alpha}+\gamma]$, $\delta(t, \alpha)$ has at least two zeros in $(0, b(\alpha))$. Set

$$\hat{b} = \inf\{b(\alpha) \mid \alpha \in (\bar{\alpha}, \bar{\alpha}+\gamma)\}, \tag{5.1}$$

Since $b(\alpha) \to \infty$ as $\alpha \to \bar{\alpha}$, $\alpha > \bar{\alpha}$, $\hat{b}$ can be attained by some $\hat{\alpha} \in (\bar{\alpha}, \bar{\alpha}+\gamma]$, namely, $b(\hat{\alpha}) = \hat{b}$. We can suppose $b(\alpha) > \hat{b}$ for all $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

Since $\delta(t, \hat{\alpha})$ has at least two zeros in $(0, \hat{b})$, $u(t, \alpha)$ and $u(t, \hat{\alpha})$ have at least two intersection points in $(0, \hat{b})$ for some $\bar{\alpha} < \alpha < \hat{\alpha}$. Because of $b(\alpha) > \hat{b}$, there is a third intersection point of these two solutions in $(0, \hat{b})$. As we decrease α, the second and third intersection points cannot approach the first intersection point or $(\hat{b}, 0)$, nor can they coincide. But $u(t, \bar{\alpha})$ has only one intersection point with $u(t, \hat{\alpha})$ in $(0, \hat{b})$. This is true if γ is sufficiently small by the proof of Lemma 4.4. for if necessary, we may choose another number less than γ. Thus we obtain a contradiction. The proof is completed.

LEMMA 5.2. *If $u(t, \alpha)$, $\alpha > 0$ is a crossing solution, then $\delta(t, \alpha)$ does not have a second zero point exactly at $b(\alpha)$.*

Proof. The proof of this lemma is similar to the proof of Lemma 4.4. We employ identities (4.5) and (4.16). Suppose to the contrary that for some $\alpha > 0$, $u(t, \alpha)$ is a crossing solution, and $\delta(t, \alpha)$ has a first zero at $\tau_\alpha \in (0, b(\alpha))$ and the second zero at $b(\alpha)$. Then

$$\begin{aligned} &\delta(t, \alpha) > 0, \qquad \text{for} \quad t \in (0, \tau_\alpha), \qquad \delta(t, \alpha) < 0 \qquad \text{for} \quad t \in (\tau_\alpha, b(\alpha)). \\ &\delta(b(\alpha), \alpha) = 0, \qquad \delta'(b(\alpha), \alpha) > 0. \end{aligned} \tag{5.2}$$

By (4.5) and (4.16) we have

$$\int_0^{b(\alpha)} t^{n-1} \delta(f(u) - uf'(u))\, dt = 0,$$

$$\int_0^{b(\alpha)} n^{n-1} \delta(3f(u) - uf'(u))\, dt = b^n(\alpha)\, u'(b(\alpha))\, \delta'(b(\alpha)) < 0.$$

Thus for any $\beta > 1$, we have

$$\int_0^{b(\alpha)} t^{n-1}\delta(\beta f(u) - uf'(u)) < 0. \tag{5.3}$$

But, similar to the proof in Lemma 4.3, we can see that, in any case, (5.3) cannot be true simultaneously for both $\beta = p > 1$ and $\beta = q > 1$, and so the proof is completed.

Now we are in a position to complete the proof of our main theorems.

Proof of Theorem 1. We simply set $\alpha^* = \bar{\alpha}$. By the definition of $\bar{\alpha}$, we see that all solutions $u(t, \alpha)$ in $0 < \alpha < \alpha^*$ are slowly decaying. By Lemma 3.1, $u(t, \alpha^*)$ is a fast decaying solution. By Lemma 3.3, $u(t, \alpha^*)$ intersects with $u(t, \alpha)$, $0 < \alpha < \alpha^*$, exactly once in $t > 0$. So (i) and (ii) of Theorem 1 are proved.

It follows from Lemma 4.4 that any solution $u(t, \alpha)$ with α slightly bigger than α^* is a crossing solution. By Lemma 5.1, there is some $\hat{\alpha} > \alpha^*$ such that $u(t, \alpha)$ is a crossing solution, and $\delta(t, \hat{\alpha})$ vanishes exactly once in $(0, b(\alpha))$, so then $\delta(b(\hat{\alpha}), \hat{\alpha}) < 0$. By differentiating $u(b(\alpha), \alpha) = 0$ with respect to α, we have

$$u'(b(\alpha), \alpha)\, b'(\alpha) + \delta(b(\alpha), \alpha) = 0.$$

So

$$b'(\alpha) = -\frac{\delta(b(\alpha), \alpha)}{u'(b(\alpha), \alpha)} < 0, \tag{5.4}$$

which shows that $b(\alpha)$ is a decreasing solution of α near $\hat{\alpha}$. As we increase α from $\hat{\alpha}$, by using a continuity argument (see [16, 17]), we can see that the number of zeros of $\delta(t, \alpha)$ cannot increase, otherwise at some time, $\delta(t, \alpha)$ has exactly one second zero at $b(\alpha)$. But this has been excluded by Lemma 5.2. If we decrease α from $\hat{\alpha}$ to α^*, the same argument works. So we conclude that for any $\alpha > \alpha^*$, $u(t, \alpha)$ is a crossing solution and $b(\alpha)$ is a strictly decreasing function. The proof is completed.

Proof of Theorem 2. In view of Theorem 1, we only need to prove the existence of solutions to problem (1.1). By the continuous dependence of solutions to initial value problems for ordinary differential equations, we see that $b(\alpha)$ tends to infinity as α approaches α^* from above. While $b(\alpha)$ tends to zero when $\alpha \to \infty$ (see [3, 9]). Thus $b(\alpha)$ ranges from 0 to ∞, and for any given ball Ω with radius $b > 0$, there is one (and only one) α such that $b(\alpha) = b$. The proof is completed.

REFERENCES

1. H. Berestycki and P.-L. Lions, Non-linear scalar field equation. I, Existence of a ground state, II, Existence of infinitely many solutions, *Arch. Rational Mech. Anal.* **82** (1983), 313–375.
2. J. G. Berryman and C. J. Holland, Stability of separable solutions for fast diffusions, *Arch. Rational Mech. Anal.* **74** (1980), 379–388.
3. A. Castro and A. Kurepa, Infinitely many radially symmetric solutions to a superlinear Dirichlet problem in a ball, *Proc. Amer. Math. Soc.* **101** (1987), 57–64.
4. Chiun-Chuan Chen and Chang-Shou Lin, Uniqueness of the ground state solution of $\Delta u+f(u)=0$ in R^n, $n\geqslant 3$, *Comm. Partial Differential Equations* **16** (1991), 1549–1572.
5. C. V. Coffman, On the positive solutions of boundary value problems for a class of nonlinear differential equations, *J. Differential Equations* **3** (1967), 92–111.
6. C. V. Coffman and M. Marcus, Existence and uniqueness results for semi-linear Dirichlet problems in Annuli, *Arch. Rational Mech. Anal.* **108** (1989), 293–307.
7. Y. Deng and D. Cao, Uniqueness of the positive solution for singular non-linear boundary value problems, *Syst. Sci. Math. Sci.* **6** (1993), 25–31.
8. B. Gidas, W.-M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.
9. R. Kajikiya, Radially symmetric solutions of semilinear elliptic equations, existence and Sobolev estimates, *Hiroshima Math. J.* **21** (1991), 111–161.
10. R. Kajikiya, Infinitely many radially symmetric solutions of certain semi-linear elliptic equations, *Hiroshima Math. J.* **21** (1991), 557–585.
11. R. Kajikiya, Existence and asymptotic behavior of nodal solutions for semi-linear elliptic equations, *J. Differential Equations* **106** (1993), 238–256.
12. N. Kawano, E. Yanagida, and S. Yotsutani, Structure theorems for positive radial solutions to $\Delta u+K(|x|)\,u^p=0$ in R^n, *Funkcial. Ekvac.* **36** (1993), 557–579.
13. M. K. Kwong, Uniqueness of positive solutions of $\Delta u-u+u^p=0$ in R^n, *Arch. Rational Mech. Anal.* **105** (1989), 243–266.
14. M. K. Kwong and Y. Li, Uniqueness of radial solutions of semilinear elliptic equations, *Trans. Amer. Math. Soc.* **333** (1992), 339–363.
15. M. K. Kwong, J. B. McLeod, L. A. Peletier, and W. C. Troy, On ground state solutions of $-\Delta u=u^p-u^q$, *J. Differential Equations* **95** (1992), 218–239.
16. M. K. Kwong and L. Zhang, Uniqueness of the positive solution of $\Delta u+f(u)=0$ in an annulus, *Differential Integral Equations* **4** (1991), 583–596.
17. J. B. McLeod, Uniqueness of positive radial solutions of $\Delta u+f(u)=0$ in R^n, II, *Trans. Amer. Math. Soc.* **339** (1993), 495–505.
18. J. B. McLeod and J. Serrin, Uniqueness of positive radial solutions of $\Delta u+f(u)=0$ in R^n, *Arch. Rational Mech. Anal.* **99** (1987), 115–145.
19. J. B. McLeod, W. C. Troy, and F. B. Weissler, Radial solutions of $\Delta u+f(u)=0$ with prescribed numbers of zeros, *J. Differential Equations* **83** (1990), 368–378.
20. K. Nagasaki, Radial solutions for $\Delta u+|x|^l\,|u|^{p-1}\,u=0$ on the unit ball in R^n, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **36** (1989), 211–232.
21. W.-M. Ni, Uniqueness of solutions of nonlinear Dirichlet problems, *J. Differential Equations* **50** (1983), 289–304.
22. W.-M. Ni and R. D. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u+f(u,r)=0$, *Comm. Pure Appl. Math.* **38** (1985), 69–108.
23. L. A. Peletier and J. Serrin, Uniqueness of positive solutions of semilinear equations in R^n, *Arch. Rational Mech. Anal.* **81** (1983), 181–197.
24. L. A. Peletier and J. Serrin, Uniqueness of non-negative solutions of semilinear equations in R^n, *J. Differential Equations* **61** (1986), 380–397.

25. S. I. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet. Math.* **5** (1965), 1408–1411.
26. M. Struwe, Superlinear elliptic boundary value problems with rotational symmetry, *Arch. Math.* **39** (1982), 233–240.
27. W. C. Troy, Bounded solutions of $\Delta u + |u|^{p-1} u - |u|^{q-1} u = 0$ in the supercritical case, *SIAM J. Math. Anal.* **21** (1990), 1326–1334.
28. E. Yanagida, Structure of positive radial solutions of Matukuma's equation, *Japan J. Indust. Appl. Math.* **8** (1991), 165–173.